Bounds on eigenvalues of the Laplacian for certain classes of closed hyperbolic 3-manifolds

by

Nina Juliana White

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Doctoral Committee:

Professor Richard D. Canary, Co-Chair
Associate Professor Juan Souto, Co-Chair, University of British Columbia
Professor Ralf J. Spatzier
Professor Alejandro Uribe-Ahumada
Assistant Professor Vilma M. Mesa

To Juliette.

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CHAPTER I

Introduction

Recall that the Laplace-Beltrami operator $\Delta = -\operatorname{div} \nabla$ on a closed Riemannian manifold M is a self-adjoint linear operator defined on $H^2(M)$, the Sobelev space of twice weakly differentiable L^2-integrable functions on M. The set of values λ satisfying $\Delta f = \lambda f$ is positive and discrete and can be ordered

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

The main result of this thesis relates the kth eigenvalue of the Laplace-Beltrami operator of certain closed hyperbolic 3-manifolds to their volume. Biringer and Souto proved in [5] that, given $\varepsilon, c, \delta > 0$, there exist only finitely many isometry classes of hyperbolic 3–manifolds M with $\operatorname{inj}(M) \geq \varepsilon$, $\operatorname{rank}(\pi_1(M)) \leq c$, and $\lambda_1 > \delta$. Due to Wang's finiteness theorem for ε-thick hyperbolic 3-manifolds, this is equivalent to saying that, with injectivity radius and rank bounds in place, $\lambda_1(M) \to 0$ as $\operatorname{vol}(M) \to \infty$. The following theorem implies their result and additionally provides a precise asymptotic statement.

Theorem I.1. *For every $\varepsilon > 0$, $c, k \in \mathbb{N}$ there exists $\Omega(\varepsilon, c, k)$ such that, if M is a closed, ε-thick hyperbolic 3-manifold with $\operatorname{rank}(\pi_1(M)) < c$, then*

$$\frac{1}{\Omega \operatorname{vol}^2(M)} \leq \lambda_k(M) \leq \frac{\Omega}{\operatorname{vol}^2(M)}$$

where $\lambda_k(M)$ is the kth positive eigenvalue of the Laplace operator on M.

Recall that the *rank* of a group is the minimal number of elements in a generating set. The *injectivity radius* of a closed hyperbolic manifold M is half the length of the smallest essential closed curve in M. If the injectivity radius of a manifold M is greater than ε, we say the manifold is ε-*thick*.

Since we will refer to the hypotheses of Theorem I.1 repeatedly, from now on we will say that,

> *Given ε and c, a closed hyperbolic 3-manifold M satisfies $(*)$ if the following holds:*

$$\boxed{(*) \qquad\qquad \operatorname{inj}(M) \geq \varepsilon \quad \textit{and} \quad \operatorname{rank}(\pi_1(M)) \leq c}$$

This notation represses ε and c, but will always be used in the context of *fixed* ε and c.

Notice that for suitable choices of ε and c, there are infinitely-many examples of manifolds satisfying $(*)$; think, for instance, of cyclic covers of a manifold fibering over the circle.

We discuss briefly the role of the various constants in the hypotheses of the theorem. The lower bound in the theorem was proved by Schoen [29] and does not depend on k, c, or ε. The upper bound, however, definitively depends on c: Long-Lubotzky-Reid showed in [23] that every closed hyperbolic 3-manifold has a cofinal family of covers $\{M_i\}$ with Property τ. For any such family, $\operatorname{vol}(M_i) \to \infty$ and $\liminf \lambda_1(M_i) > 0$. The upper bound also definitively depends on k because $\lambda_k(M) \to \infty$. The proof of the upper bound presented here certainly depends on the injectivity radius bound ε, and the author highly suspects it's necessary for the theorem to hold. The proof given also heavily relies on the structure provided by hyperbolicity. In general, some kind of curvature bound is necessary because Colbois and Dodziuk proved the following theorem in [17]:

Theorem I.2 (Colboi-Dodziuk [17])**.** *Every compact manifold M^n with $n \geq 3$ admits metrics g of volume one with arbitrarily large $\lambda_1(M_g)$.*

To prove the upper bound in Theorem I.1 we use the "max" part of the Minimax Theorem. Loosely speaking, we can bound λ_k from above by constructing test functions with bounded Rayleigh quotient.

With the end of constructing these functions, we introduce a theorem of Biringer and Souto (see Theorem V.1). Namely, for M satisfying $(*)$, there is a topological decomposition of M into pieces including large-diameter product regions. Further, "deep enough" into these product regions, the geometry is approximately like that of a simply-degenerate end of an infinite-volume hyperbolic 3-manifold. We strengthen their theorem and show that these product regions can be fibered by surfaces of bounded geometry (see Theorem V.2). Finally, using this fact, we construct step-like test functions with bounded Rayleigh quotients.

Before jumping into the next chapter, we give a brief outline of this thesis. Chapters II – IV are background. In Chapter II we review some hyperbolic geometry. In particular, we recall two concepts that will be essential to understanding the decomposition theorems in Chapter V: convergence of hyperbolic 3-manifolds, and (simply-degenerate) ends of hyperbolic 3-manifolds. In Chapter III we recount related results in spectral theory. In Chapter IV we briefly go over some topological facts we need about 3-manifolds and, in particular, product regions. The proof of Theorem I.1 is spread out over the last two chapters. Chapter V is the densest chapter, culminating in the proof of the strengthening of Biringer and Souto's theorem described above. Finally, in Chapter VI, we use this stronger decomposition theorem to construct test functions with bounded Rayleigh quotient and prove Theorem I.1.

CHAPTER II

Preliminaries: Hyperbolic Geometry

2.1 Basics

For complete expositions on hyperbolic 3-manifolds see [2], [24], [26]. After recalling some basic definitions and theorems from hyperbolic geometry, we will examine two concepts we will need for the proof in Chapter V, namely convergence of hyperbolic manifolds (§2.2) and degenerate ends of hyperbolic 3-manifolds (§2.3).

A hyperbolic 3-manifold M is a complete, connected Riemannian 3-manifold with constant curvature -1. We will only consider orientable hyperbolic 3-manifolds in this thesis. In this case M is isometric to a quotient $\mathbb{H}^3/\Gamma$, where $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^3)$ is discrete and torsion-free. We will often additionally require that Γ is nonabelian so as not to worry about overly simple examples where theorems tend to fail (see, for example Theorems II.3 and II.4). We can identify $\mathrm{Isom}^+(\mathbb{H}^3)$ with $\mathrm{PSL}(2,\mathbb{C})$ through its action on boundary of $\mathbb{H}^3$ (see [26, Theorem 1.8]). However, conjugate subgroups of $\mathrm{PSL}(2,\mathbb{C})$ yield isometric quotients. To resolve this ambiguity we can instead consider *pointed* hyperbolic 3-manifolds (M,ω), in which M has a specified basepoint x and a specified orthonormal frame ω at $T_x(M)$. Fix a baseframe $\omega_{\mathbb{H}^3}$ once and forever. Now, given (M,ω) we can determine $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ with $M = \mathbb{H}^3/\Gamma$ uniquely by requiring that ω lifts to $\omega_{\mathbb{H}^3}$. Given (M,ω), denote the unique such group

$\Gamma_{(M,\omega)}$. In summary, pointed hyperbolic 3-manifolds are in one-to-one correspondence with discrete, torsion-free subgroups of $\mathrm{PSL}(2,\mathbb{C})$ (see [2, Proposition E.1.9]), and, as we will see in §2.2.2, these spaces are homeomorphic when given the geometric topology.

2.1.1 Thick-thin decomposition

The *injectivity radius* at a point x in M is defined to be half the length of the shortest essential closed curve through x and denoted $\mathrm{inj}_M(x)$.

Given $\varepsilon > 0$, we define the ε*–thick* part of M to be

$$M_{[\varepsilon,\infty)} := \{x \in M \mid \mathrm{inj}_M(x) \geq \varepsilon\}$$

and the ε*–thin part* of M to be

$$M_{(0,\varepsilon]} := \overline{\{x \in M \mid \mathrm{inj}_M(x) < \varepsilon\}}$$

If ε is small enough, then components of $M_{(0,\varepsilon]}$ have a very simple description. Let's first introduce some terminology. Using the coordinate identification of $\mathbb{H}^3$ with the upper half-space, a *horoball* in $\mathbb{H}^3$ is a region isometric to

$$H_c = \{(x,y,t) \in \mathbb{H}^3 \,|\, t > c\}$$

for some $c > 0$. A *banana-shaped region* in $\mathbb{H}^3$ is a region isometric to

$$B_c = \{p \in \mathbb{H}^3 \,|\, d(p,(0,0,t)) \leq c, \text{ for some } c \in (0,\infty)\},$$

for some $c \geq 0$. Note that a version of the following theorem holds for general dimension n, but we restrict ourselves to dimension 3.

Theorem II.1 (Thick-Thin Decomposition)**.** *There exists ε' (the Margulis constant in dimension 3, see [2, Chapter D]) such that if $\varepsilon < \varepsilon'$, then a component of $M_{(0,\varepsilon]}$ is isometric to exactly one of the following:*

(i) B_c/Γ, *where* $\Gamma = \langle\gamma\rangle$ *and* γ *is a loxodromic element fixing the* z*-axis. Topologically these are solid tori and will be called* Margulis tubes.

(ii) H_c/Γ, *where* $\Gamma = \langle\gamma\rangle$ *and* γ *is a parabolic element fixing* $\infty \in \overline{C}$. *This is called a* rank-1 cusp.

(iii) H_c/Γ, *where* $\Gamma = \langle\gamma_1, \gamma_2\rangle$ *such that* γ_1, γ_2 *both fix* $\infty \in \overline{C}$ *and do not generate a cyclic group. This is called a* rank-2 cusp.

2.1.2 Mostow-Prasad Rigidity

Mostow originally proved this important rigidity theorem for closed hyperbolic manifolds only [27]. Prasad generalized this to finite volume hyperbolic manifolds in [28].

Theorem II.2. *For* $n \geq 3$, *two finite-volume hyperbolic* n*-manifolds are isometric if and only if they have isomorphic fundamental groups.*

We will refer to this result in the discussion of Wang's finiteness theorem (see Theorem II.6).

2.2 Convergence

2.2.1 Algebraic convergence

Let Γ be a finitely-generated, torsion-free, nonabelian Kleinian group with fixed generating set $\Gamma = \langle\gamma_1, \ldots, \gamma_k\rangle$. Let $\mathcal{D}(\Gamma)$ denote the set of discrete faithful representations of Γ. Since a homomorphism $\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ is determined by the image of the generators, $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{C}))$ sits inside $\mathrm{PSL}(2, \mathbb{C})^k$ and thus inherits its topology. We say the $\{\rho_i\} \subset \mathcal{D}(\Gamma)$ converge *algebraically* to $\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ if they converge in $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{C}))$. Jørgensen showed $\mathcal{D}(\Gamma)$ is closed with respect to algebraic convergence, that is $\rho \in \mathcal{D}(\Gamma)$.

Theorem II.3 (Jørgensen [21]). *Let Γ be a non-elementary Kleinian group. Given a sequence of faithful representations $\rho_i : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ such that each $\rho_i(\Gamma)$ is discrete and they have limit $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$, then ρ is also faithful and $\rho(\Gamma)$ is also discrete.*

2.2.2 Geometric convergence

A sequence of subgroups $\Gamma_i \subset \mathrm{PSL}(2,\mathbb{C})$ converges *geometrically* to Γ if they converge respect to the Chabauty topology on closed subsets of $\mathrm{PSL}(2,\mathbb{C})$. Recall that if $\{C_i\}$ is a sequence of closed subsets in $\mathrm{PSL}(2,\mathbb{C})$, then $C_i \to C$ in the Chabauty topology if and only if the following two conditions hold:

(i) If $x \in C$, then there exists a sequence $\{x_i\} \in C_i$ such that $x_i \to x$ in $\mathrm{PSL}(2,\mathbb{C})$.

(ii) If $\{x_i\} \subset \mathrm{PSL}(2,\mathbb{C})$ converge to x, then $x \in C$.

At the level of quotient manifolds, we say a sequence of pointed manifolds $\{(M_i,\omega_i)\}$ *converges geometrically* to (M,ω) if there exists sequences $\{R_i\} \to \infty$, $\{L_i\} \to 1$, such that for each i there is an L_i-bi-Lipschitz embedding $N_{R_i}(x) \to M_i$ taking $\omega \mapsto \omega_i$, where $N_{R_i}(x_i)$ is the R_i-neighborhood of x_i, ω_i a frame for $T_{x_i}M_i$. This can also be called *Gromov-Hausdorff convergence.* We often repress $\{R_i\}$ and $\{L_i\}$ and describe this colloquially by saying that the M_i *approximate* M for large enough i.

These two definitions of geometric convergence at the level of groups and on the level of pointed manifolds are equivalent in the following sense:

Theorem II.4 (see [26, Theorem 7.7] for a proof). *Fix a baseframe $\omega_{\mathbb{H}^3}$ in $\mathbb{H}^3$. If a sequence Γ_n of Kleinian groups converge geometrically to a non-elementary (i.e. nonabelian) Kleinian group G, then the hyperbolic 3-manifolds $(\mathbb{H}^3/\Gamma_n, \pi_n(\omega_{\mathbb{H}^3}))$ converge geometrically to $(\mathbb{H}^3/\Gamma, \pi(\omega_{\mathbb{H}^3}))$, where π and π_n denote the standard quotient* *maps. Conversely, if the hyperbolic 3-manifolds (M_n, ω_n) converge geometrically to (M, ω), then $\{\Gamma_{(M_n,\omega_n)}\}$ converges geometrically to $\Gamma_{(M,\omega)}$.*

Remark. In many cases it will be sufficient to forget about the baseframe and talk about convergence of pointed manifolds (this time without baseframe): $\{(M_i, x_i)\}$, $x_i \in M_i$.

When we have lower bounds on injectivity radius, we have the following useful theorem.

Proposition II.5 (See [26, Proposition 7.8] for a proof.)**.** *If a sequence of pointed hyperbolic 3-manifolds (M_i, x_i) has a positive lower bound on the injectivity radii at the basepoints x_i, then some subsequence converges geometrically.*

This means that as long as basepoints are chosen in the thick part, sequences will always converge up to subsequence.

A consequence of the above proposition and Mostow's rigidity is Wang's finiteness theorem. In three dimensions this says:

Theorem II.6 (Wang's finiteness theorem for dimension 3)**.** *Given $\varepsilon, V > 0$, there exist only finitely-many closed hyperbolic 3-manifolds with injectivity radius greater than ε and volume less than V.*

Proof. The maximum diameter of an ε-thick hyperbolic 3-manifold M with $\mathrm{vol}(M) < V$ is

$$d = \frac{V}{\omega_3(\varepsilon)}(2\varepsilon),$$

where $\omega_3(\varepsilon)$ is the volume of a ball in $\mathbb{H}^3$ with radius ε.

An infinite sequence $\{M_i\}$ of ε-thick hyperbolic 3-manifolds converges up to subsequence for any choice of basepoint by Proposition II.5. However, because they are all of bounded diameter, the manifolds in the tail of the sequence are eventually homeomorphic to the limit. However, by Mostow's rigidity, this means they are all isometric. □

2.2.3 Smooth geometric convergence

There is another notion of geometric convergence on quotient manifolds we need to introduce: smooth geometric convergence. This definition is more complicated to state, and we only need it once, in the proof of Theorem V.2, so we've saved it for last. *A priori* it will sound much stronger than Gromov-Hausdorff convergence, but it is actually topologically equivalent to the Gromov-Hausdorff topology on baseframed hyperbolic 3-manifolds described above; both are equivalent to the Chabauty topology on torsion-free Kleinian groups, see [2, Theorem E.1.13] and [26, Theorem 7.7].

We will start with what at first sounds like a straight-forward definition and define ambiguous terms as we go along. Feel free to take this first attempt as the definition, if you want to avoid the details that follow. A sequence of pointed hyperbolic 3-manifolds (M_i, ω_i) converge *smoothly geometrically* to $(M_\infty, \omega_\infty)$ if for every compact set $K \subset M_\infty$ containing the baseframe, for large enough i there exist smooth embeddings

$$\varphi_i : K \to M_i$$

taking ω_∞ to ω_i and converging smoothly to an isometric embedding.

This sounds nice, but we need to explain more precisely what it means to "converge smoothly to an isometric embedding." Fix once and for all a baseframe for $\mathbb{H}^3$, call it $\omega_{\mathbb{H}^3}$. The φ_i converge smoothly to an isometric embedding if the lifts $\widetilde{\varphi}_i : \widetilde{K} \to \mathbb{H}^3$ converge smoothly to the identity, where the choice of lift takes ω_∞ to $\omega_{\mathbb{H}^3}$. This also sounds nice, but we need to explain more precisely what it means to "converge smoothly to the identity" for maps from subsets of $\mathbb{H}^3$ to $\mathbb{H}^3$. Heuristically, the $\widetilde{\varphi}_i$ converge smoothly to the identity if they converge pointwise and if their derivatives converge. However, we don't require that all derivatives converge at the same rate; we have tolerance for higher derivatives converging more slowly. Also note that to compare derivatives in the first place we need to use parallel transport.

This formal description of smooth convergence is lifted almost entirely from [2]. Let K be a compact subset of $\mathbb{H}^3$ and let f, g be maps from a neighborhood of K to $\mathbb{H}^3$. Let $P_{x,y} : T_y\mathbb{H}^3 \to T_x\mathbb{H}^3$ denote parallel transport along the geodesic joining x and y. Let d_p denote the pth differential. Then for each $p \geq 1$, $z \in K$, we can compare pth derivatives of f and g at z as follows:

$$P_{f(z),z} \circ d_p f(z) - P_{f(z),z} \circ d_p g(z)$$

We define the distance between f and g on K to be:

$$\begin{aligned} D(f,g)_K := &\max_{z\in K} d(f(z), g(z)) \\ &+ \sum_{p=1}^{\infty} 2^{-p} \cdot \left(1 \wedge \max_{z\in K} \left|\left|P_{f(z),z} \circ d_p f(z) - P_{f(z),z} \circ d_p g(z)\right|\right|\right), \end{aligned}$$

where $\wedge$ means "minimum." Then $\{f_i\}$ converges smoothly to f on K if $D(f_i, f)_K \to 0$. When U is an open set in $\mathbb{H}^3$, then f_i converges smoothly to f on U if $D(f_i, f)_K \to 0$ for every compact subset $K \subset U$.

2.2.4 Strong convergence

Given an algebraically convergent sequence $\{\rho_i\} \to \rho$ in $\mathcal{D}(\Gamma)$, assume the $\rho_i(\Gamma)$ converge geometrically to a group $G \subset \mathrm{PSL}(2, \mathbb{C})$. If $G = \rho(\Gamma)$ we call the convergence *strong*. However, this won't always be the case. While it's clear that $\rho(\Gamma) \subset G$, Jørgensen [20] found examples where the geometric limit was strictly larger than the algebraic limit.

For an overview of some sufficient conditions for strong convergence see, for example, [26, §7.3]. A specific result we will need to reference is the following:

Theorem II.7 (Brock-Bromberg-Canary-Minsky [8]). *Let N be a compact 3-manifold and let $\{\rho_i\} \subset \mathcal{D}(\pi_1(N))$ converge algebraically to ρ. If every parabolic element of $\rho(\pi_1(N))$ lies in a rank two free abelian subgroup, then $\{\rho_i\}$ converges strongly to ρ.*

2.3 Ends of tame hyperbolic 3-manifolds (without cusps)

2.3.1 Tameness

A hyperbolic 3-manifold is *tame* if it is homeomorphic to the interior of a compact manifold (perhaps with boundary). Marden conjectured in [25] that every hyperbolic 3-manifold with finitely-generated fundamental group is tame. This was recently proved independently by Agol [1] and Calegari-Gabai [12].

Given an end $\mathcal{E}$ in a tame manifold, a neighborhood of $\mathcal{E}$ is homeomorphic to $S \times [0, \infty)$, where S is a closed surface coming from the corresponding boundary component of the compact core. Recall that the compact core of a 3-manifold M is a compact submanifold C such that the inclusion $C \hookrightarrow M$ is a homotopy equivalence. Scott proved that every 3-manifold with finitely-generated fundamental group admits a compact core [30].

Remark. When discussing tameness and the classification of ends of hyperbolic 3-manifolds it is convenient to impose a uniform lower bound on injectivity radius. Unless we specifically mention otherwise, we will restrict ourselves to this setting.

Canary proved in [14] that tameness can also be described by the geometry of the relative ends. (See the remark at the end of this chapter for a description of the difference between ends and relative ends. When a lower bound on the injectivity radius is in place, they are the same.) A manifold is tame if and only if its (relative) ends are either *geometrically finite* or *simply degenerate.*

2.3.2 Geometrically Finite ends

Given a hyperbolic 3-manifold M, let $CC(M)$ denote the *convex core* of M. Recall that the convex core of M is the smallest convex set in M whose inclusion $CC(M) \hookrightarrow M$ is a homotopy equivalence.

An end $\mathcal{E}$ of M is *geometrically finite* if its there is a neighborhood of $\mathcal{E}$ disjoint from $CC(M)$. One defining property of such ends is that they have exponentially expanding geometry.

2.3.3 Simply degenerate ends

In light of Canary's result [14] we could define simply degenerate ends in a tame manifolds to be ends which are not geometrically finite. However a more precise description of their geometry warrants our attention. In particular, an end is simply degenerate if and only if it has simplicial hyperbolic surfaces homotopic to the boundary at infinity exiting the end. Recall the definition of a simplicial hyperbolic surface, the use of which was pioneered by Bonahon in [6] in order to examine the geometry of simply-degenerate ends.

Definition II.8. Given S a surface of genus $g \geq 2$, τ a triangulation of S, and $\varphi : S \to M$ an immersion, a triple (S, τ, φ) is a *simplicial hyperbolic surface* in a hyperbolic 3-manifold M if the following properties are satisfied:

(i) The 1-simplices in τ are mapped to geodesics in M;

(ii) The 2-simplices in τ are mapped to totally geodesic triangles in M;

(iii) When the metric from M is pulled back to S, the angle measure around each vertex measures $\geq 2\pi$.

The angle condition in the definition ensures that the surfaces have negative curvature and hence bounded area by Gauss-Bonnet. In particular, we could also describe simply-degenerate ends as ends with surfaces of bounded area in the right homotopy class exiting the end. This is in direct contrast to the exponential growth of geometrically finite ends. Canary's Filling Theorem [15] showed further that, not only do simplicial hyperbolic surfaces exit a simply-degenerate end, but they fill a simply degenerate end in the following sense:

Theorem II.9 (Filling Theorem, Canary [15])**.** *Let M be a topologically tame hyperbolic 3-manifold without cusps and $\mathcal{E}$ a geometrically infinite end. Then $\mathcal{E}$ has a neighborhood U homeomorphic to $S \times [0, \infty)$ such that every point in a subneighborhood $\hat{U} = S \times [k, \infty)$ is in the image of some convenient simplicial hyperbolic surface $f_x : S \to U$ such that f_x is (properly) homotopic (within U) to $S \times \{0\}$.*

Jorgensen demonstrated the first examples of ends that were not geometrically finite, [22]. His examples were cyclic covers of closed manifolds fibering over the circle. This is generally a good example to picture when thinking about degenerate ends.

When a lower bound on injectivity radius is in place, we can describe simply degenerate ends of tame manifolds in the following way:

Corollary II.10. *In the presence of a lower bound on the injectivity radius, an end of a tame hyperbolic 3-manifold is simply degenerate if and only if there exist surfaces in the correct homotopy class with bounded diameter exiting the end.*

Proof. Let E be a simply degenerate end of a tame hyperbolic 3-manifold M. Then E has simplicial hyperbolic surfaces $S \to M$ in the right homotopy class exiting the end. Negative curvature of these simplicial hyperbolic surfaces ensures they all have area at most $A = 2\pi|\chi(S)|$ by Gauss-Bonnet. Further, because the surfaces are ε-thick, they all have diameter at most:

$$d = \frac{A}{\omega_2(\varepsilon)}(2\varepsilon),$$

where $\omega_2(\varepsilon)$ is the area of a ball in $\mathbb{H}^2$ of radius ε.

Conversely, a geometrically finite end cannot have surfaces in the correct homotopy class of bounded diameter exiting the end. To see this, first recall that the nearest point retraction map onto the convex core gives rise to coordinates $S \times [0, \infty)$ on the geometrically finite end, with respect to which the metric on the end is quasi-isometric to $\cosh^2(t)ds_S^2 + dt^2$. The level surfaces with respect to these coordinates have exponentially expanding geometry. Also, the projection of the end onto any level set is area-reducing. Any sequence of surfaces exiting the end would project onto deeper and deeper level surfaces of the end via area-reducing nearest point retraction maps. In the presence of a lower bound on injectivity radius, this precludes the possibility that the original surfaces had bounded diameter. $\square$

2.4 Doubly-Degenerate Manifolds

A *doubly-degenerate manifold* refers to a hyperbolic 3-manifold without parabolics homeomorphic to $\Sigma \times \mathbb{R}$ with two degenerate ends, where Σ is a closed surface. In the decomposition theorem of Biringer and Souto (see TheoremV.1) the geometry of manifolds satisfying $(*)$ is understood through comparison to doubly-degenerate manifolds. Here we examine doubly-degenerate manifolds in more detail in order utilize their geometry in Chapter V, in which we strengthen Biringer-Souto's decomposition theorem.

We're now ready to prove the following well-known fact:

Fact II.11. *Let $\mathcal{M}_{g,\varepsilon}$ be the set of ε-thick, doubly-degenerate manifolds homeomorphic to $\Sigma_g \times \mathbb{R}$, where Σ_g is a closed surface of genus g. Then $\mathcal{M}_{g,\varepsilon}$ is compact with respect to geometric convergence.*

We expand upon a proof given in [3, Proposition 3.2].

Proof. Given a pointed sequence $(M_i, x_i) \in \mathcal{M}_{\varepsilon,g}$, we first show that it converges algebraically (after choosing appropriate markings) and geometrically up to subsequence. Next we show that the respective limits are the same. Lastly we combine algebraic convergence and tameness to show that the limit is homeomorphic to $\Sigma_g \times \mathbb{R}$ and combine geometric convergence with the Filling Theorem to show the limit is doubly-degenerate.

By Canary's Filling Theorem [15] we know that for each i there is a simplicial hyperbolic surface $f_i : S_i \to M_i$ whose image contains x_i. Fix a finite generating set $X = \langle \gamma_1, \dots, \gamma_k \rangle \subset \pi_1(\Sigma_g)$. We can use the following lemma to carefully choose markings on the simplicial hyperbolic surfaces (S_i, τ_i, f_i).

> **Lemma II.12** (Short Markings, Biringer [3])**.** *Given ε, g and a finite generating set $X \subset \pi_1(\Sigma_g)$, there exists ℓ such that whenever $f : \Sigma_g \to M$ is an ε-thick simplicial hyperbolic surface of genus g and $p \in \Sigma_g$, there is an isomorphism: $\pi_1(\Sigma_g) \to \pi_1(\Sigma_g, p)$ such that the image of each element of X can be represented by a loop based at p of length less than ℓ (where the metric on Σ_g is pulled back from M).*

Recall that the f_i are length preserving, so by composing an isomorphism from the above lemma with f_i, we get markings ρ_i on (M_i, x_i) such that for all i, j, $\rho_i(\gamma_j)$ lies in bounded subset of $\mathrm{PSL}(2, \mathbb{C})$ consisting of elements with translation length at the origin in the interval $[\varepsilon, \ell]$. Therefore, up to subsequence, $\{\rho_i\}$ converges algebraically to $\rho \in \mathcal{D}(\pi_1(\Sigma_g))$.

Additionally, because the manifolds are all ε-thick, the groups $\rho_i(\pi_1(\Sigma_g))$ have a geometric limit G and because ε-thickness precludes the existence of parabolics in the limit, Theorem II.7 ensures convergence is strong.

By tameness, the geometric limit (M_∞, x_∞) is also homeomorphic to $\Sigma_g \times \mathbb{R}$. We have only to show that it has two degenerate ends. In light of Corollary II.10, we want to find a diameter bound $d > 0$ such that if $K \subset M_\infty$ is a compact subset containing x_∞ and separating the ends of M_∞, we can find a surface in each component of $M_\infty - K$ homotopic to M_∞ with diameter less than d. This is easy: we have M_i-approximations of M_∞ of diameter much larger than $\text{diam}(K)$. We can pull back simplicial hyperbolic surfaces from the M_i (which exist where we want them to by Canary's Filling Theorem) to get immersed surfaces in $M_\infty - K$ with bounded diameter. This is true for a compact exhaustion of M_∞, so by Corollary II.10, the manifold is doubly-degenerate. $\square$

Remark: Full disclosure on injectivity radius bounds

We've simplified our discussion of ends by assuming our manifolds have injectivity radius bounded away from 0. Two things can happen without this assumption.

First, without the lower bound on injectivity radius we may have cusps. In this case, fix ε less than the Margulis constant and let M^0_ε be M without the ε-thin components corresponding to rank-1 and rank-2 cusps. When classifying ends of manifolds with cusps we only consider the ends of M^0_ε, called the *relative ends.*

The other case we've not yet considered is less intuitive. Bonahon and Otal showed the existence of simply-degenerate ends with arbitrarily short closed geodesics [7]. In this case the simplicial hyperbolic surfaces filling the end still have bounded area, but their diameter grows without bound. In particular, Corollary II.10 does not hold in this case. Also, such examples do not resemble cyclic covers of manifolds fibering over the circle.

CHAPTER III

Related Results in Spectral Theory

3.1 Basics

The Laplace-Beltrami operator on a closed Riemannian manifold M is a linear operator on $C^\infty(M)$ defined to be:

$$\Delta f = -\operatorname{div}(\nabla f)$$

Recall that $H^2(M)$ is the Sobelev space of L^2-integrable functions on M that are twice weakly differentiable. With respect to the $H^2(M)$-norm, Δ is a bounded linear operator on $C^\infty(M)$ and has a unique continuous extension to $H^2(M)$. Thus, more generally, Δ is a linear operator $\Delta : H^2(M) \to L^2(M)$.

Green's Identity states that for $f, g \in C^\infty(M)$:

$$\int_M f \Delta g \, dM = \int_M \langle \nabla f, \nabla g \rangle \, dM \tag{3.1}$$

It follows that Δ is self-adjoint and thus has real, non-negative eigenvalues. The set of eigenvalues:

$$\{\lambda \in \mathbb{R} \mid \exists f \in H^2(M) \text{ such that } \Delta f = \lambda f\}$$

is called the *spectrum* of the Laplace-Beltrami operator on M. For convenience, we will often refer to the Laplace-Beltrami operator as simply the *Laplacian* and to the spectrum of the Laplacian on M as simply *the spectrum of M*.

The spectral theorem describes the spectrum and eigenfunctions as follows:

Theorem III.1 (Spectral Theorem, as given in [11])**.** *Let M be a compact connected Riemannian manifold without boundary. The eigenvalue problem*

$$\Delta\varphi = \lambda\varphi$$

has a complete orthonormal system of C^∞-eigenfunctions $\varphi_0, \varphi_1, \ldots$ in $L^2(M)$ with corresponding eigenvalues $\lambda_0, \lambda_1, \ldots$ such that

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots, \lambda_n \to \infty \text{ as } n \to \infty.$$

Green's Theorem and the Spectral Theorem combine to give the Minimax theorem [11] (the key to the proof of Theorem I.1) in which λ_k is bounded above by the maximum Rayleigh quotient of a family of $k+1$ functions with disjoint support. More precisely:

Theorem III.2 (Minimax Theorem)**.** *Let $h_0, \ldots, h_k \in C^\infty(M)$ be positive functions which satisfy*

$$\mathrm{vol}(\mathrm{supp}\, h_i \cap \mathrm{supp}\, h_j) = 0$$

for every $0 \leq i < j \leq k$. Then

$$\lambda_k(M) \leq \max_{0 \leq j \leq k} R(h_j),$$

where $R(h_j)$ is the Rayleigh quotient:

$$R(h_j) = \frac{\int_M ||\nabla h_j||^2 \, dM}{\int_M h_j^2 dM}.$$

We give an idea of the general proof by proving the statement for $k = 1$. The complete proof can be found in [11].

Proof of case $k = 1$. First note that $\lambda_0 = 0$ because, letting $f \equiv 1$,

$$\Delta f = -\operatorname{div}(\nabla f) = 0.$$

Let $g_0, g_1 \in C^\infty(M)$ be positive functions such that $\operatorname{supp}(g_0) \cap \operatorname{supp}(g_1)$ has measure 0. By the spectral theorem g_0, g_1 can be expressed in terms of the Hilbert basis of eigenfunctions as:

$$g_0 = \sum_{i=0}^{\infty} \alpha_i \varphi_i,\ \alpha_i \in \mathbb{C}$$

$$g_1 = \sum_{i=0}^{\infty} \beta_i \varphi_i,\ \beta_i \in \mathbb{C}.$$

Case 1: $\alpha_0 = \beta_0 = 0$

Applying the Laplacian to g_0 gives us: $\Delta g_0 = \sum_{i=1}^{\infty} \lambda_i \alpha_i \varphi_i$. By Green's Theorem we have:

$$\begin{aligned} \int_M g_0 \Delta g_0 dM &= \int_M \langle \nabla g_0, \nabla g_0 \rangle dM \\ \left\langle \sum_{i=1}^{\infty} \alpha_i \varphi_i, \sum_{i=1}^{\infty} \lambda_i \alpha_i \varphi_i \right\rangle_{L^2} &= \int_M ||\nabla g_0||^2 \, dM \\ \lambda_1 \langle g_0, g_0 \rangle_{L^2} &\leq \int_M ||\nabla g_0||^2 \, dM \\ \lambda_1 &\leq \frac{\int_M ||\nabla g_0||^2 \, dM}{\int_M g_0^2 dM} = R(g_0) \end{aligned}$$

The analogous inequality holds for g_1, so the statement is proved.

Case 2: Without loss of generality $\beta_0 \neq 0$

Let $f = g_0 - \frac{\alpha_0}{\beta_0} g_1$. Then when f is expressed in terms of the Hilbert basis of eigenfunctions, the coefficient of φ_0 is 0. Namely,

$$f = \sum_{i=1}^{\infty} \left(\alpha_i - \frac{\alpha_0}{\beta_0} \beta_i \right) \varphi_i.$$

Applying the calculation in Case 1 and using the fact that g_0 and g_1 have disjoint support, we get:

$$\lambda_1 \le \frac{\int_M ||\nabla f||^2 \, dM}{\int_M f^2 dM} = \frac{\int_{\mathrm{supp}\, g_0} ||\nabla g_0||^2 \, dM + \frac{\alpha_0^2}{\beta_0^2} \int_{\mathrm{supp}\, g_1} ||\nabla g_1||^2 \, dM}{\int_{\mathrm{supp}\, g_0} g_0^2 dM + \frac{\alpha_0^2}{\beta_0^2} \int_{\mathrm{supp}\, g_1} g_1^2 dM}$$

To finish the proof we have only to apply the following cute fact[1] for positive numbers a, b, c, d:

$$\frac{a+b}{c+d} \le \max \left\{ \frac{a}{c}, \frac{b}{d} \right\}$$

□

So far we've assumed every manifold is *closed*, that is, compact without boundary. When we drop either of those conditions, the above definitions and results do not necessarily hold. Specifically, if we consider compact Riemannian manifolds with boundary, we must specify boundary conditions on functions $f \in H^2(M)$ when solving the differential equation $\Delta f - \lambda f = 0$. In this thesis we won't explore this case.

On the other hand, we will occasionally discuss results about non-compact manifolds to point out analogies to closed manifolds. The Laplace operator and its spectrum in this setting are defined differently.

When M is noncompact, we must mention that the Laplacian only acts on compactly supported functions and the spectrum, defined as

$$\mathrm{spec}(M) = \{\lambda \in \mathbb{C} \,|\, \Delta - \lambda I \text{ is not invertible}\},$$

consists of more than just the eigenvalues of Δ. Further, the spectrum is not discrete: it consists of two parts: the *essential spectrum*, accumulation points of the spectrum and eigenvalues with infinite multiplicity; and the *discrete spectrum*, isolated eigenvalues with finite multiplicity. In the face of this complexity, we restrict

[1]To prove this fact, assume $\frac{a+b}{c+d} > \frac{a}{c}, \frac{b}{d}$. This leads to two contradictory statements: $bc > da$ and $da > bc$.

our attention to the smallest non-trivial eigenvalue $\lambda_0(M)$, whose definition we can take to be:

$$\lambda_0 = \inf_{f \in C_0^\infty(M)} \frac{\int_M ||\nabla f||^2 \, dM}{\int_M f^2 dM} \tag{3.2}$$

(Note that in the case that M is closed, $\lambda_1(M)$ can be defined the same way with the infimum instead running over $f \in C^\infty(M)$ with $\int f dM = 0$.)

3.2 Cheeger's Isoperimetric Constant and $\lambda_1(M)$

Although we prove our main theorem using the Minimax theorem, many previous results relating the spectrum of M to its volume are proved using the well-known relationship between the $\lambda_1(M)$ and Cheeger's isoperimetric constant $h(M)$.

Recall that when M is a 3-dimensional Riemannian manifold, Cheeger's isoperimetric constant is defined to be:

$$h(M) = \inf_N \frac{\text{area}(\partial N)}{\min\{\text{vol}(N), \text{vol}(M - N)\}},$$

where the infimum is taken over all 3-dimensional submanifolds $N \subset M$ (not necessarily connected) with Lipschitz boundary. When M has infinite volume, we require N to be compact to ensure the division makes sense. Cheeger showed the following relationship between $\lambda_1(M)$ and $h(M)$.

Theorem III.3 (Cheeger's Inequality, [16])**.** *When M is a closed Riemannian manifold,*

$$\lambda_1(M) \geq h^2(M)/4. \tag{3.3}$$

In the case that M has curvature bounded from below, Buser found a lower bound for $\lambda_1(M)$ in terms of $h(M)$.

Theorem III.4 (Buser's Inequality, [10]). *If the Ricci curvature of a closed Riemannian manifold M^n is bounded below by $-(n-1)\delta^2$ ($\delta \geq 0$) then:*

$$\lambda_1(M) \leq 2\delta(n-1)h(M) + 10h^2(M). \tag{3.4}$$

In the case that M is non-compact, versions of Cheeger's and Buser's bounds still hold. Cheeger's result holds as is for non-compact infinite-volume manifolds without boundary if we replace $\lambda_1(M)$ with $\lambda_0(M)$ in the statement of the theorem. (See definition of $\lambda_0(M)$ in (3.2).) Buser's theorem can be revised in the non-compact setting as follows:

Theorem III.5 (Buser's Inequality, [10]). *If M is complete, non-compact and has Ricci curvature bounded below by $-(n-1)\delta^2$ ($\delta \geq 0$), then:*

$$\lambda_0(M) \leq c\delta h(M), \tag{3.5}$$

where c depends only on the dimension of M.

3.3 Volume and $\lambda_1(M)$ for closed manifolds

Recall that the main result of this thesis states that, for closed hyperbolic 3-manifolds M satisfying $(*)$, $\Omega^{-1}\,\mathrm{vol}^{-2}(M) \leq \lambda_k(M) \leq \Omega\,\mathrm{vol}^{-2}(M)$, where Ω depends only on k, $\mathrm{rank}(\pi_1(M))$ and $\mathrm{inj}(M)$.

As we proceed to summarize analogous results, we avoid excessive primes and subscripts by abusing notation slightly and overusing the constant Ω. Keep in mind that these instances of Ω are only defined within the scope of the theorem statements and are not related.

Schoen found a lower bound on λ_1 in [29]. Because $\lambda_k \geq \lambda_1$, his result gives rise to the lower bound in our main theorem. Schoen's result states that:

Theorem III.6 (Schoen [29])**.** *If the sectional curvatures of M satisfy the inequality $-1 \leq K_M \leq -\kappa^2$ for some $\kappa \in (0,1)$ and if $n \geq 3$, then the first eigenvalue $\lambda_1(M)$ satisfies*

$$\lambda_1(M) \geq \min\left\{\frac{(n-1)^2\kappa^2}{4}, \frac{\delta_n}{\mathrm{vol}^2(M)}\right\} \geq \frac{\delta'_{n,\kappa}}{\mathrm{vol}^2(M)}$$

where $\delta_n = 4^-1\omega_{n-1}^2[\varepsilon_n e^{\varepsilon_n(1-\kappa)}]^{2n-2}$, $\varepsilon_n = 4^{-(n+3)}$, $\omega_n =$volume of the unit ball in $\mathbb{R}^n$, and

$$\delta'_{n,\kappa} = (n-1)^2 4^{-1}\kappa^2\omega_n^2\varepsilon_n^{2n}.$$

In the case of hyperbolic 3-manifolds, this says $\lambda_1(M) \geq \pi^2 4^{-34} 3^{-2}\,\mathrm{vol}^{-2}(M)$.

Remark. For $n = 2$, on the other hand, no such lower bound in terms of volume can exist. A genus-g hyperbolic surface has area $4\pi(g-1)$. Further, given a fixed genus g we can find genus-g surfaces with arbitrarily small Cheeger constant (and thus $\lambda_1(M)$ by Buser's inequality) by pinching a non-trivial separating curve. See Theorem III.9 for an alternate bound that holds in this case.

Schoen proves this result using Cheeger's inequality. In particular, though he never states it explicitly, he proves the following lemma on the way to his main result:

$$h(M) \geq \Omega\,\mathrm{vol}^{-1}(M), \tag{3.6}$$

where Ω depends only on the curvature and and dimension of M. We will use this fact in §3.5 below.

Theorem I.1 is an improvement on an upper bound for $\lambda_1(M)$ in terms of $\mathrm{vol}(M)$ proved by Biringer and Souto. Their theorem, also in the context of M satisfying $(*)$, states that:

Theorem III.7. *For every $\varepsilon, \delta, c > 0$, there are only finitely-many isometry classes* *of hyperbolic 3-manifolds M with injectivity radius* $\operatorname{inj}(M) \leq \varepsilon$*, first eigenvalue* $\lambda_1(M) \geq \delta$ *and* $\operatorname{rank}(\pi_1(M)) \leq c$*.*

In their paper, Biringer and Souto use Buser's inequality to first prove the following statement: for M satisfying $(*)$, $\lambda_1(M) \lesssim \operatorname{vol}^{-1}(M)$. Then they apply Wang's Finiteness Theorem to get Theorem III.7 above. In light of Theorem I.1, their result is not optimal. In fact, as we will show in §3.5, under the $(*)$ hypotheses Buser's upper bound cannot recover the asymptotics of Theorem I.1.

3.4 Volume and $\lambda_0(M)$ for open manifolds

The upper and lower bounds discussed in the last section have analogies on certain classes of non-compact hyperbolic manifolds.

We will start with lower bounds on $\lambda_0(M)$. Unlike Schoen's lower bound for closed manifolds, finding such bounds in the case of non-compact manifolds heavily utilizes hyperbolic structure.

For example, Dodziuk's and Randol's generalization of Schoen's bound (see Theorem III.6) to finite-volume hyperbolic manifolds utilizes thick-thin decomposition by estimating λ_0 separately on the thick and thin parts of M using the Minimax theorem.

Theorem III.8 (Dodziuk-Randol [18])**.** *If M^n is a finite-volume hyperbolic n-manifold, $n \geq 3$, then there exists $\Omega(n)$ depending only on n such that*

$$\lambda_0 \geq \Omega(n) \operatorname{vol}^{-2}(M)$$

As mentioned before, such a bound clearly fails for $n = 2$. In this case, the following theorem holds instead.

Theorem III.9 (Dodziuk-Randol [18])**.** *Given $V > 0$, there exists $\Omega(V) > 0$ such that if M is a hyperbolic surface with volume V, then*

$$\lambda_0(M) > \Omega(V)L,$$

where L is the minimal length of a separating chain of simple closed geodesics.

In the case in which the volume is not finite, Burger and Canary proved the following lower bound on $\lambda_0(M)$ for geometrically finite hyperbolic 3-manifolds.

Theorem III.10 (Burger-Canary [9])**.** *For all $n \geq 3$, there exists a constant $\Omega(n) > 0$ such that if M is an infinite volume, geometrically finite hyperbolic n-manifold, then*

$$\lambda_0(M) \geq \Omega(n)\,\mathrm{vol}^{-2}(C_1(M)),$$

where $\mathrm{vol}(C_1(M))$ denotes the volume of the neighborhood $C_1(M)$ of radius one of the convex core.

As for upper bounds, Canary proved the following theorem in [13].

Theorem III.11. *Let M be an infinite volume, topologically tame hyperbolic 3-manifold. Then $\lambda_0(M) = 0$ if and only if M is not geometrically finite. Moreover, there exists a constant Ω such that if M is geometrically finite, then*

$$\lambda_0(N) \leq \Omega \frac{|\chi(\partial C(M))|}{vol(C(M))}$$

where $C(M)$ denotes the the convex core of M.

This theorem can be seen as an infinite-volume analog to Theorem I.1. The role that $|\chi(\partial C(N))|$ plays in Canary's theorem is similar to the role of the bound on $\mathrm{rank}(\pi_1(M))$ in the hypotheses of Theorem I.1, although the relationship in that case is not necessarily linear. Canary's proof bounds $h(M)$ and applies Buser's inequality for infinite-volume manifolds.

Let M be a complete, connected, hyperbolic 3-manifold. Then the various bounds on $\lambda_1(M)$ (resp. $\lambda_0(M)$) in terms of volume and $h(M)$ can be summarized as follows:

	Upper Bounds	Lower Bounds
Compact	Buser: (R. curv $> -\kappa$) $\implies \lambda_1(M) \lesssim h^2(M) + h(M)$ Biringer-Souto: $(*) \implies \lambda_1(M) \lesssim \mathrm{vol}^{-1}(M)$ W.: $(*) \implies \lambda_1(M) \lesssim \mathrm{vol}^{-2}(M)$	Cheeger: $\lambda_1(M) \gtrsim h^2(M)$ Schoen: $(0 >$ S. curv $> \kappa) \implies \lambda_1(M) \gtrsim \mathrm{vol}^{-2}(M)$
Finite Vol.		Dodziuk-Randol: $\lambda_0(M) \gtrsim \mathrm{vol}^{-2}(M)$
Geo. Finite	Buser: (curv $> \kappa$) $\implies \lesssim h(M)$ Canary: $\lambda_0(M) \lesssim \lvert\chi(\partial C(M))\rvert\, \mathrm{vol}^{-1}(C(M))$	Cheeger: $\lambda_0(M) \gtrsim h^2(M)$ Burger-Canary: $\lambda_0(M) \gtrsim \mathrm{vol}^{-2}(C(M))$

3.5 A new result relating volume and $h(M)$

In the case that M is a closed hyperbolic 3-manifold satisfying $(*)$, an explicit relationship between volume, $h(M)$, and $\lambda_1(M)$ arises by combining the various bounds discussed in §3.2 - §3.3.

From (3.3) and Theorem I.1 we obtain that, under appropriate bounds on rank and injectivity radius, $h(M) \leq \frac{2\sqrt{\Omega}}{\mathrm{vol}(M)}$. Combining this with Schoen's result in (3.6) we get a new corollary:

Corollary III.12. *Given ε and c, there exists $\Omega' > 1$ such that if M is a closed hyperbolic 3-manifold as in the statement of Theorem I.1, then*

$$\frac{1}{\Omega' \,\mathrm{vol}(M)} \leq h(M) \leq \frac{\Omega'}{\mathrm{vol}(M)}. \qquad \square$$

In light of this corollary, Buser's inequality yields at best $\lambda_1(M) \lesssim \mathrm{vol}^{-1}(M)$ under the appropriate hypotheses, a strictly weaker result than Theorem I.1.

CHAPTER IV

Tools: Waldhausen's Cobordism Theorem and Corollaries

In order to effectively apply Theorem V.1, we need to recall some facts about 3-manifold topology. In particular, a well-known corollary of Waldhausen's Cobordism Theorem will be essential. In this section we justify certain 3-manifold facts we will need in Chapter V.

Recall that an open (respectively closed) *product region* is a 3-manifold homeomorphic to $\Sigma \times (0,1)$ (respectively $\Sigma \times [0,1]$), where Σ is a closed orientable surface.

Any embedded surface in a product region homotopic to a boundary component will be called a *fiber*. It follows from Waldhausen's Theorem [19, Corollary 13.7] that two disjoint fibers in a product region bound a product region.

We say an embedded surface in a 3-manifold is *separating* if its complement is disconnected. Given a 3–manifold with two boundary components, a separating surface in the interior *separates the boundary components* if every arc connecting the boundary components intersects the surface. Otherwise, the separating surface bounds a compact region not meeting the boundary.

The following topological facts about product regions are certainly known to experts. We include brief explanations for the reader's benefit.

Let $T = \Sigma \times I$ be a product region of genus g. Let Σ' be a closed embedded surface in T. We can view Σ' as an element of $H_2(T;\mathbb{Q})$. Recall that there exists a non-degenerate bilinear pairing $H_1(T,\partial T;\mathbb{Q}) \times H_2(T;\mathbb{Q})$ given by intersection number. Because $\dim H_1(T,\partial T;\mathbb{Q}) = \dim H_2(T;\mathbb{Q}) = 1$, the following dichotomy holds: either Σ' has nonzero algebraic intersection number with the arc $\{x\} \times [0,1]$, for some $x \in \Sigma$, in which case it separates the boundary components of T, or Σ is null-homologous and bounds a compact region not meeting the boundary. In particular:

Fact IV.1. *There are no embedded, non-separating surfaces in the interior of T.*

Continuing with the same notation, let p be the projection:

$$p : T = \Sigma \times I \to \Sigma \text{ given by } (x,t) \mapsto x.$$

If Σ' has genus less than g then the restriction $p|_{\Sigma'}$ cannot be π_1-injective. In other words Σ' is compressible in T [19] and by repeatedly compressing, we reduce genus and end up with a sphere, which must be null-homologous because the universal cover of T is contractible. Being null-homologous, Σ' bounds a compact region not meeting the boundary. In summary:

Fact IV.2. *If Σ' is a closed embedded orientable surface in T with genus smaller than g, then Σ' bounds a compact region. Conversely, if Σ' separates the boundary components of T, then the genus of Σ' is greater than or equal to that of T.*

If Σ' is an embedded surface of genus g and separates the boundary components of T then Σ' is non-trivial in $H_2(T;\mathbb{Q})$. As such, $p|_{\Sigma'}$ must be π_1-injective, otherwise Σ' would be compressible, therefore homologous to a smaller-genus surface, therefore null-homologous by Fact IV.2. Because $p|_{\Sigma'}$ is π_1-injective, it is homotopic to a covering map and, since the genera are the same, homotopic to a bijection. In other words:

Fact IV.3. *If Σ' and T have the same genus and Σ' separates the boundary components of T, then Σ' is a fiber in T.*

CHAPTER V

A Decomposition Theorem

The proof of Theorem I.1 follows quite easily once we establish a useful decomposition theorem for manifolds satisfying $(*)$. We start by introducing a theorem of Biringer and Souto which gives us a decomposition of M satisfying $(*)$ into pieces including large-diameter product regions. The bulk of the work in this chapter is proving a stronger version of their theorem, Theorem V.2, which shows that those product regions can be fibered by surfaces of bounded geometry. From there, the decomposition theorem, Corollary V.7, follows immediately.

5.1 Biringer-Souto Theorem

Biringer and Souto proved the following theorem in [4].

Theorem V.1 (Biringer-Souto)**.** *For every $\varepsilon, c > 0$ there exists a finite set $\{Q_1, \ldots, Q_s\}$ of compact Riemannian 3-manifolds (perhaps with boundary) and $L, r > 1$ such that if M is a closed hyperbolic 3-manifold with*

$$(*) \qquad \mathrm{inj}(M) \geq \varepsilon \textit{ and } \mathrm{rank}(\pi_1(M)) \leq c$$

then M contains a compact submanifold $\overline{M}$ with the following properties:

1. *$\overline{M}$ has at most r components, each one of them L-bi-Lipschitz equivalent to one of the Q_i,*

2. *Each component of $M - \overline{M}$ is homeomorphic to $\Sigma_g \times \mathbb{R}$, where Σ_g is a closed orientable surface of genus g.*

Further, given any pairwise distinct pointed sequence (x_i, M_i) such that each M_i satisfies $(*)$ and $d(x_i, \overline{M_i}) \to \infty$, some subsequence converges geometrically to (x_∞, M_∞), where M_∞ is an ε-thick doubly-degenerate manifold.

We refer to $\overline{M}$ as the *tiny manifold*. The first part of the theorem tells us that components of the complement of the tiny manifold are all product regions, a purely topological statement. A helpful picture to illustrate the theorem is shown in Figure 5.1.

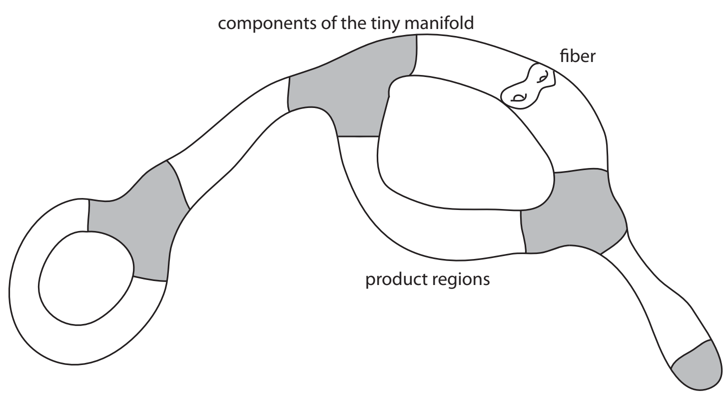

Figure 5.1: A picture illustrating the Biringer-Souto Theorem. The "tiny manifold" $\overline{M}$ is shown in grey. The white parts are product regions.

Let's point out what we don't know immediately from Theorem V.1. The first part of the theorem is purely topological, and does not give geometric information about the product regions. The second part of the theorem is geometric, but is only a pointwise statement. A neighborhood of a point far enough away from the tiny manifold will look like a product region neighborhood in a doubly-degenerate manifold of genus g, but a priori this local structure may not be compatible with the topological structure of the larger product region component— in particular, a priori it may not be the same genus. Chapter II outlines the basis of hyperbolic geometry and defines the important terms in Theorem V.1.

5.2 A Stronger Version

We improve this loose point-wise geometric statement by showing that we can actually fiber these product regions with surfaces of uniformly bounded geometry. To this end, we frequently refer to a fixed metric family of "canonical" product regions $\{S_g \times [-1,1] \,|\, S_g \in \mathcal{S}\}$ endowed with the product metric, where $\mathcal{S} = \{S_g\}_{g=2}^{\infty}$ is a fixed family of hyperbolic structures, one for each genus $g \geq 2$.

The main result of this section is the following:

Theorem V.2. *For every $\varepsilon, c > 0$ there exists a finite set $\{Q_1, \ldots, Q_s\}$ of compact Riemannian 3-manifolds (perhaps with boundary) and $L, r, G > 1$ such that if M is a closed hyperbolic 3-manifold with*

$$(*) \qquad \operatorname{inj}(M) \geq \varepsilon \text{ and } \operatorname{rank} \pi_1(M) \leq c$$

then M contains a compact submanifold $\widetilde{M}$ with the following properties:

1. *$\widetilde{M}$ has at most r components, each one of them L-bi-Lipschitz equivalent to one of the Q_i.*
2. *Each component T of $M - \widetilde{M}$ is homeomorphic to $\Sigma_g \times \mathbb{R}$, where Σ_g is a closed orientable surface of genus $g < G$.*
3. *Further, there is a foliation of T by genus-g surfaces compatible with the product*

structure such that for any point $x \in T$, there is a leaf-preserving L-bi-Lipschitz embedding

$$S_g \times [-1,1] \to M, \quad S_g \in \mathcal{S}$$

with x in the image of the 0-fiber.

5.3 Proof of Theorem V.2

We start by showing the existence of G in the statement of Theorem V.2.

Lemma V.3. *There exists G such that given M_i satisfying $(*)$ and $x_i \in M_i$ with $d(x_i, \overline{M_i}) \to \infty$ then the doubly-degenerate geometric limit M_∞ guaranteed by Theorem V.1 has genus at most G.*

Proof. Suppose no such bound on genus existed. Then for every $g \in \mathbb{N}$, there exists a sequence of pointed manifolds $({}^g x_i, {}^g M_i)$ satisfying $(*)$ with $d({}^g x, \overline{{}^g M_i}) \to \infty$ and converging to a doubly degenerate manifold $({}^g x_\infty, {}^g M_\infty)$ of genus g. Because each ${}^g M_\infty$ is ε–thick we can assume that up to subsequence $({}^g x_\infty, {}^g M_\infty)$ converges to some (x_∞, M_∞). Notice that (x_∞, M_∞) is also the geometric limit of some diagonal sequence $({}^g x_i, {}^g M_i)$, and, as such, is an ε-thick doubly-degenerate manifold of some genus G by Theorem V.1.

Fix an embedded fiber $\Sigma \subset M_\infty$ through x_∞. For $g > G$ large enough, we can approximate $\Sigma \subset M_\infty$ by ${}^g\Sigma \subset {}^g M_\infty$. Because ${}^g\Sigma$ has genus G and ${}^g M_\infty$ has genus $g > G$, ${}^g\Sigma$ bounds a compact region ${}^g K \subset {}^g M_\infty$ by Fact IV.2. Note that $\operatorname{diam}({}^g K) \to \infty$ because otherwise ${}^g K$ would lie in an approximation of M_∞. See Figure 5.2.

Similarly, for each g we can choose index $i(g)$ large enough to approximate ${}^g K \subset {}^g M_\infty$ by ${}^g K_{i(g)} \subset {}^g M_{i(g)}$. Note also that $\operatorname{diam}({}^g K_{i(g)}) \to \infty$.

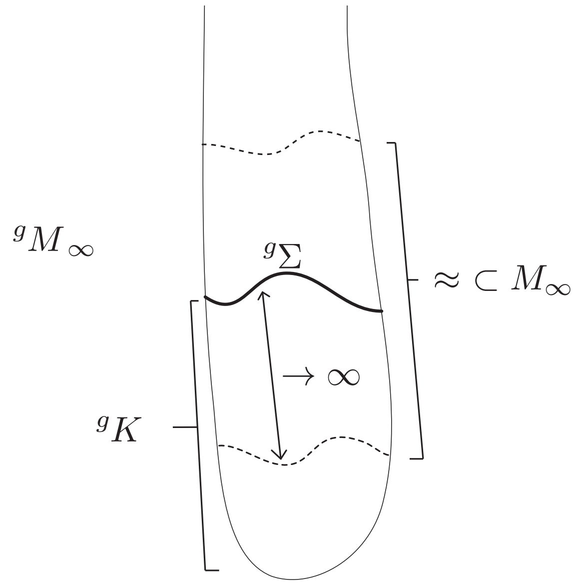

Figure 5.2: As $g \to \infty$, diameter of ^gK must also go to ∞.

We are now ready to shed this elaborate notation: rename the sequence $({}^gx_{i(g)}, {}^gM_{i(g)})$ with one index j. Likewise, let $K_j = {}^gK_{i(g)}$ and let $\Sigma_j = \partial K_j$.

Let $x'_j \in K_j$ be a point furthest from Σ_j. Note that $d(x'_j, \overline{M_j}) \to \infty$ so Theorem V.1 applies and up to subsequence we have limit $(x'_j, M_j) \to (x'_\infty, M'_\infty)$, where M'_∞ is doubly-degenerate of some genus G'.

Fix an embedded fiber $\Sigma' \subset M'_\infty$ through x'_∞. Also fix an infinite arc $\gamma : \mathbb{R} \to M'_\infty$ exiting both ends. Note that γ necessarily has algebraic intersection number 1 with Σ'.

Let $\Sigma'_j \subset M_j$ be the approximation of Σ' that exists for large enough j. We briefly outline the rest of the proof: working in the approximations M_j, we use approximations of γ to find closed curves intersecting Σ'_j exactly once—a contradiction, since Σ'_j must have disconnected complement by Fact IV.1.

The closed curves will come from concatenating the γ-approximations with geodesic segments connecting to $\Sigma_j = \partial K_j$—such segments can't backtrack and go back through Σ'_j because the regions of approximation have very large diameters and x'_j is the furthest point from $\Sigma_j = \partial K_j$.

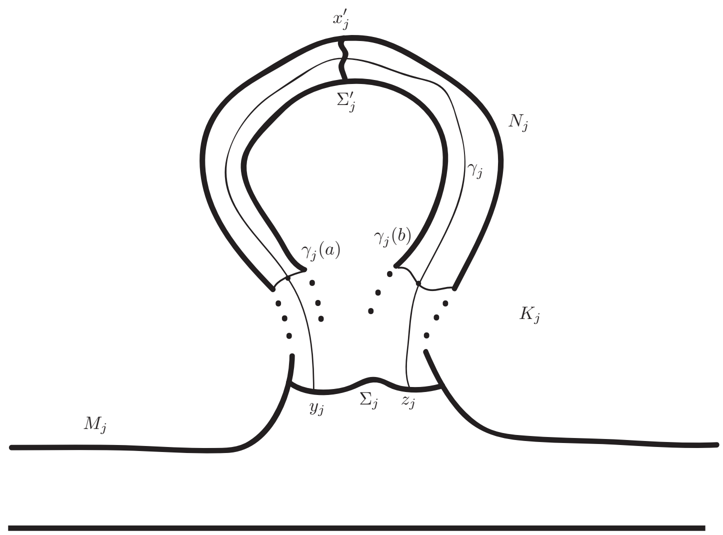

Figure 5.3: Σ'_j is an embedded nonseparating surface, a contradiction.

To help with the following nitty-gritty details, see Figure 5.3. We have increasingly large approximations $N_j \subset M_j$ of M'_∞. Because $\text{diam}(K_j) \to \infty$, large enough j allow us to impose the simultaneous restrictions: $\text{diam}(N_j) >> \text{diam}(\Sigma'_j)$ and $N_j \subset K_j$.

Note that N_j may have more than two boundary components but that the boundary components fall into two classes depending on which component of $N_j - \Sigma'_j$ they belong to. The distance between these two classes of boundary components is at least d_j, where

$$d_j = \text{diam}(N_j)/2 - \text{diam}(\Sigma'_j)$$

and thus $\lim_{j\to\infty} d_j = \infty$.

We are now ready to construct our closed curve intersecting Σ'_j exactly once. Let $\gamma_j : [a, b] \to N_j$ be the unique component of the approximation of γ with image intersecting Σ'_j. (We need to specify a particular component because we may have other small segments near ∂N_j.) Concatenate γ_j with geodesics segments $[\gamma_j(a), y_j]$, $[\gamma_j(b), z_j]$, such that $y_j, z_j \in \Sigma_j$ are the points closest to $\gamma_j(a)$ and $\gamma_j(b)$ respectively, recalling that $\Sigma_j = \partial K_j$. Finally concatenate with any arc in Σ_j connecting y_j to z_j. See Figure 5.3 for an illustration. We claim that this closed curve still intersects Σ'_j exactly once, showing that the complement to Σ'_j is connected, contradicting Fact IV.1.

We have only to show that $[\gamma_j(a), y_j]$ and $[\gamma_j(b), z_j]$ do no intersect Σ'_j. This is easy, for recall that $d_j >> \operatorname{diam}(\Sigma'_j)$. Without loss of generality suppose $[\gamma(a), y_j]$ intersected Σ'_j, then by the triangle inequality $d(\gamma_j(a), \Sigma_j) > d(x'_j, \Sigma_j)$, contradicting our initial choice of x'_j.

□

From now on let G be the constant provided by Lemma V.3.

Notice that in the course of the above proof we showed the following fact:

Fact V.4. *If a sequence of metric product regions (T_i, x_i) converge geometrically to a doubly-degenerate manifold (M_∞, x_∞), then given a fiber of M_∞ through x_∞, the approximating image of the fiber in T_i separates the boundary components for all sufficiently large i.*

Recall from Fact II.11 that the space of ε-thick doubly-degenerate manifolds of genus g is compact with respect to geometric convergence. It follows that the space of ε-thick doubly-degenerate manifolds of genus $g \leq G$ is also compact with respect to geometric convergence. Recall that we fixed a family of hyperbolic surfaces $\mathcal{S} = \{S_g\}_{g=2}^{\infty}$. Compactness yields uniformity in the following sense:

(DD1) Given $\varepsilon > 0$, $G \geq 2$, there exists $L = L(\varepsilon, G)$ such that if E is ε-thick and doubly-degenerate of genus $g \leq G$, and $x \in E$ then there exists an L-bi-Lipschitz embedding $\varphi : S_g \times [-1,1] \to E$, $S_g \in \mathcal{S}$, such that $x \in \varphi(S_g \times \{0\})$.

The next statement follows from the preceding statement letting

$$D(\varepsilon, G) = L(\varepsilon, G) \max_{2 \leq g \leq G} \{\mathrm{diam}(S_g) \,|\, S_g \in \mathcal{S}\},$$

but we will frequently need this specific formulation.

(DD2) Given $\varepsilon > 0$, $G \geq 2$, there exists $D = D(\varepsilon, G)$ such that if E is ε-thick and doubly-degenerate of genus $g \leq G$, and $x \in E$, there is an embedded fiber through x of diameter less than D.

If E an ε-thick, doubly-degenerate product region of genus $g \leq G$, then (DD2), Waldhausen's theorem, and the triangle inequality combine to give the following obvious (but useful) statement:

(DD3) Let B be a ball of radius R in E. Then B contains a product region of width $w \geq 2R - 4D(\varepsilon, G)$. That is, large-radius balls in E contain comparably large-width product regions.

Recall that the *width* of a product region is the shortest distance between its boundary components.

We give now a slightly refined version of the geometric content of Theorem V.1 incorporating the genus bound we just proved:

Corollary V.5. *For every $\varepsilon, c > 0$, $R, L > 1$, there exists $\delta(\varepsilon, c, R, L)$ such that if M satisfies $(*)$ and $x \in M$ such that $d(x, \overline{M}) > \delta$, then there is a product region $T_x \subset M - \overline{M}$ such that*

(i) *$N_R(x) \subset T_x$;*

(ii) *there exists an L-bi-Lipschitz embedding $f : T_x \to E$, where E is doubly degenerate of genus $g \leq G(\varepsilon, c)$, such that f is homotopic to a fiber in E;*

(iii) *T_x separates the boundary components of P_x, where P_x is the component of $M - \overline{M}$ containing x.*

Proof. Using the limit statement in Theorem V.1, suppose no such δ exists. Then there exists a sequence of closed hyperbolic 3-manifolds (x_i, M_i) satisfying $(*)$ with $d(x_i, \overline{M_i}) \to \infty)$ such that for all i no such approximation near x_i exists. However, by Theorem V.1 there is doubly-degenerate manifold M_∞ such that $(x_i, M_i) \to (x_\infty, M_\infty)$. Recall that (DD3) ensures that this convergent sequence not only approximates a large ball in M_∞, but a large-width compact core in M_∞, a contradiction to our assumption. And by Fact V.4 they eventually separate.

□

At this point we'd like to improve Corollary V.5 in two ways. First, the above corollary only describes local behavior; we'd like to have compatibility of the embeddings described above. That is, we want to know that the embeddings $T_x \to E$ and $T_{x'} \to E$ are homotopic for different points x, x' in the same component of $M - \overline{M}$ (far enough away from $\overline{M}$). Second, we'd like to compare the product regions T_x to concrete product regions instead of *some* doubly degenerate manifold E.

Recall that we fixed a family of hyperbolic surfaces $\mathcal{S} = \{S_g\}_{g=2}^\infty$ and associated product regions $\{S_g \times [-1,1] \,|\, S_g \in \mathcal{S}\}$ endowed with the product metric. We say a metric product region T in an ambient manifold M is *locally fibered by genus-g surfaces of L-bounded geometry* if for every $x \in T$ there exists an L-bi-Lipschitz embedding $g_x : S_g \times [-1,1] \to M$ homotopic to a fiber in T such that x lies in the image of the 0-fiber.

Lemma V.6. *Given ε, c, let M satisfy $(*)$. Then there exists $w, G, \delta > 0$, $L > 1$ such that, given a component $P \subset M - \overline{M}$ of width greater than w, there exists a* *submanifold $T \subset P$ with $P \subset N_\delta(T)$ such that T is a product region of some genus $g \leq G$. Further, T is locally fibered by genus-g surfaces of L-bounded geometry.*

Proof. Let G be as in Lemma V.3. Roughly speaking, we choose a finite set of points $\{x_i\} \subset P$ far enough away from ∂P to use Corollary V.5 and find associated large-diameter product regions T_{x_i} with union $T = \cup_i T_{x_i} \subset P$. We then use topological facts to show that the T_{x_i} are all homotopic and their union T is a product region. Lastly, we show that T is locally fibered by surfaces of bounded geometry.

Step 1: Consider any component $P \subset M - \overline{M}$ of width $w > 2\delta(\varepsilon, c, R, 1.5)$, where $R = 3(L(\varepsilon, G) + D(\varepsilon, G))$, and $\delta = \delta(\varepsilon, c, R, 1.5)$ comes from Corollary V.5. Let $\gamma : [0, w] \to P$ be a geodesic segment parameterized by arc length with endpoints in distinct components of ∂P. Let $t_1 = \delta$ and let $x_1 = \gamma(t_1)$. Let T_{x_1} be a product region associated to x_1 given by Corollary V.5. For $i \geq 2$, define t_i, x_i and T_{x_i} inductively by:

$$t_i = \max_{t \in [t_{i-1}, w-\delta]} \{t \,|\, \gamma(t) \cap T_{x_{i-1}} \neq \emptyset\},$$

$$x_i = \gamma(t_i),$$

T_i is a product region associated to x_i given by Corollary V.5.

Let $T = \cup_i T_{x_i}$. Note that following the proof of Corollary V.5, (DD2) and (DD3) allow us to assume that the diameters of the components of ∂T_{x_i} are uniformly bounded above by $1.5D(\varepsilon, G)$. Thus by our choice of R, the boundary surfaces of all the T_{x_i} are pairwise disjoint. Also note that $P \subset N_{\delta+d}(T)$, where d is the maximum possible diameter of any boundary component of the tiny manifold $\partial\overline{N}$ over all N satisfying $(*)$; this number is finite because the components of the tiny manifold come from a finite family of L-bi-Lipschitz classes.

Step 2: We claim that $T_{x_1} \cup T_{x_2}$ is a product region. Note that all fibers in T_{x_1} and T_{x_2} separate the components of ∂P by Corollary V.5. From there it's not hard to see that some fiber in T_{x_2} separates the components of ∂T_{x_1} and some fiber in T_{x_2} separates the components of ∂T_{x_1}. Facts IV.2 and IV.3 combine to show that the fibers of T_{x_1} and T_{x_2} have the same genus and are thus homotopic. By Waldhausen's Theorem, $T_{x_1} \cup T_{x_2}$ is a product region. Analogously, the fibers of all the T_{x_i} are homotopic to each other and T is a product region.

Step 3: Note that (DD1) says exactly that doubly-degenerate ε-thick manifolds E are locally fibered by surfaces of bounded geometry. Since each T_{x_i} looks like a doubly-degenerate manifold, we would like to locally fiber T with surfaces of bounded geometry by simply composing the appropriate maps. However, the T_{x_i} are bi-Lipschitz equivalent to *finite* pieces of doubly-degenerate ε-thick manifolds, so we have to know that such maps "fit" near the boundary.

Given i, let $f_i : T_{x_i} \to E_i$ be a 1.5-bi-Lipschitz embedding from Corollary V.5. Given $x \in T_{x_i}$, let $x' = (f_i)^{-1}(x) \in E_i$. From (DD1) we have an $L(\varepsilon, G)$-bi-Lipschitz embedding $\varphi_{x'} : S_g \times [-1, 1] \to E_i$ with x' in the image of the 0-fiber. In particular, $\varphi_{x'}(S_g \times [-1, 1]) \subset N_{L(\varepsilon,G)+D(\varepsilon,G)}(x') \subset E_i$. Therefore, as long as $d(x, \partial T_{x_i}) \geq 1.5(L(\varepsilon, G) + D(\varepsilon, G))$, the image of $\varphi_{x'}$ will be contained in the image of f_i. In particular, $f_i^{-1} \circ \varphi_{x'} : S_g \times [-1, 1] \to T_{x_i}$ will be well-defined. Luckily, the T_{x_i} were defined in such a way to ensure this sufficient overlap. Thus a desired $1.5L(\varepsilon, G)$-bi-Lipschitz embedding exists for every point in $T = \cup_i T_{x_i}$. Figure 5.4 visualizes these overlaps.

$\square$

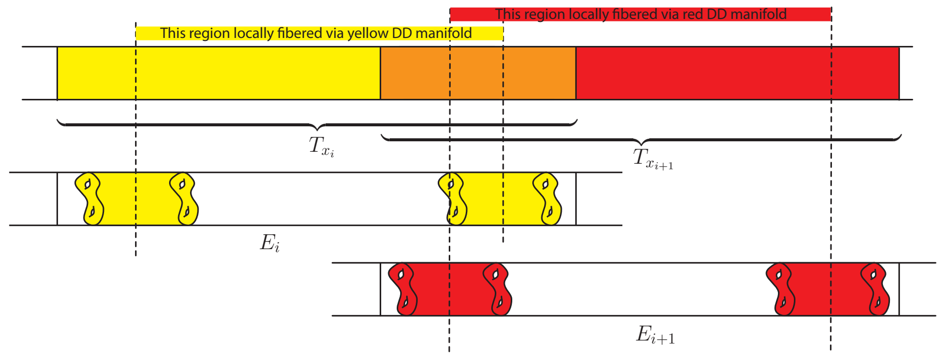

Figure 5.4: T is locally fibered by surfaces of bounded geometry.

Remark. We've used bi-Lipschitz equivalence in everything we have done so far. Recall that the definition of geometric convergence actually allows us to assume everything we have done is smooth.

We're now ready to prove Theorem V.2. We recall the theorem here for reference:

Theorem V.2. *For every $\varepsilon, c > 0$ there exists a finite set $\{Q_1, \ldots, Q_s\}$ of compact Riemannian 3-manifolds (perhaps with boundary) and $L, r, G > 1$ such that if M is a closed hyperbolic 3-manifold with*

$$(*) \qquad \operatorname{inj}(M) \geq \varepsilon \text{ and } \operatorname{rank} \pi_1(M) \leq c$$

then M contains a compact submanifold $\widetilde{M}$ with the following properties:

1. *$\widetilde{M}$ has at most r components, each one of them L-bi-Lipschitz equivalent to one of the Q_i.*

2. *Each component T of $M - \widetilde{M}$ is homeomorphic to $\Sigma_g \times \mathbb{R}$, where Σ_g is a closed orientable surface of genus $g < G$.*

3. *Further, there is a foliation of T by genus-g surfaces compatible with the product structure such that for any point $x \in T$, there is a leaf-preserving L-bi-Lipschitz*

embedding

$$S_g \times [-1,1] \to M, \quad S_g \in \mathcal{S}$$

with x in the image of the 0-fiber.

Proof. To show part (3) of the statement, it is sufficient to show the existence of a finite number of such leaf-preserving embeddings whose images cover T and intersect only on boundary fibers. (Concatenating parts of such parts gives the general result.) We do this by first showing that a "patchy" foliation exists using Lemma V.6. Then we use limits to fill in missing pieces. Lastly, we change our choice of tiny manifold while keeping the components in a finite family of L-bi-Lipschitz classes.

Step 1: Let w, L be as in Lemma V.6, let P be any component of M of width larger than w, and let $T \subset P$ be a genus-g product region as in Lemma V.6. (It's possible that no such P exists, in which case Step 3 still applies.) We will cover T with a "patchy" foliation. That is, we will find a family of L-bi-Lipschitz embeddings as in Lemma V.6 such that the distance between the images of subsequent maps is bounded both above and below.

Let w' be the width of T and let $\gamma : [0, w'] \to T$ be a geodesic connecting the boundary components of T parameterized by arc length. Let $d = \operatorname{diam}(S_g \times [-1,1])$.

Let $x_i = \gamma\left(i(Ld+1)\right)$ for $i = 0, 1, \ldots, \lfloor \frac{w'}{Ld+1} \rfloor$. By Lemma V.6, for each i we have an L-bi-Lipschitz embedding $f_i : S_g \times [-1,1] \to M$ such that x_i is in the image of the 0-fiber. Note that we have spaced the x_i so that the distance between the image of f_i and f_{i+1} is at least 1 and at most $2Ld + 1$. By Waldhausen's Theorem, the missing pieces are also product regions.

Step 2: Our goal now is to complete the patchy foliation above. Given maps f_i and

f_{i+1} from Step 1, we *fill in the gap* by finding an embedding $g : S_g \times [-1,1] \to T$ such that

(i) g is homotopic to f_i;

(ii) $\operatorname{im} g \cup \operatorname{im} f_i \cup \operatorname{im} f_{i+1}$ is a product region;

(iii) $\operatorname{vol}(\operatorname{im} g \cap \operatorname{im} f_i) = \operatorname{vol}(\operatorname{im} g \cap \operatorname{im} f_{i+1}) = 0$;

Given a particular gap, there exists *some* L' such that we can fill in the gap with an L'-bi-Lipschitz embedding $S_g \times [-1,1] \to T$. At the sake of loosing the extremities of T, suppose there is no universal choice of δ', L' such that, when distance δ' away from ∂T we can fill in gaps with L'-bi-Lipschitz embeddings. Then there exists a sequence (M_j), with associated $T_j \subset P_j$ from Lemma V.6 with

(i) a gap distance δ_j from ∂T such that

(ii) L'_j is the smallest bi-Lipschitz constant to fill that gap, and

(iii) $L'_j, \delta'_j \to \infty$.

Let x_j be any point in the gap in question and consider the limit $(x_j, M_j) \to (x_\infty, M_\infty)$ guaranteed by Theorem V.1. The gap in the limit can be filled in with an L-bi-Lipschitz map $g : S_g \times [-,1,1] \to M_\infty$ for some L. When composed with the approximating maps, this gives us a contradiction.

In summary, given a large-enough-width product region component P, there is a subset T, which, up to bounded-diameter ends, can be foliated in the desired way. We will now use T to denote that part that can be foliated.

Step 3: In what follows we will use $\overline{M}$ to refer to the tiny manifold from the statement of Theorem V.1 and $\widetilde{M}$ for our prospective tiny manifold.

Now we throw out the parts of P we weren't able to foliate and tack them on to the tiny manifold. We want to show we can choose a new set $\{Q'_1, \dots Q'_{s'}\}$ such that when M satisfies $(*)$, $\widetilde{M} = \overline{M} \cup \{P_\alpha - T_\alpha\}_\alpha$ taken over the family of components $P_\alpha \subset M - \overline{M}$, satisfies the conditions of being the tiny manifold. First note that because $\#\pi_0(M - \overline{M}) \geq \#\pi_0(M - \widetilde{M})$, we have the same bound r on the number of components in $\widetilde{M}$.

Note that there exists a uniform diameter bound d on components of $\overline{M}$. Thus any component of $\widetilde{M}$ has diameter bounded by $r(d+2\delta)$, where δ is the largest-diameter of any piece we had to cut off of the P_α to get the associated T_α.

Suppose there exists manifolds M_i satisfying $(*)$, each with some component $M_i^0 \subset \widetilde{M} \cup \{(P_i)_\alpha - (T_i)_\alpha\}_\alpha$, which together make up an infinite collection of distinct 2-bi-Lipschitz classes. Let $x_i \subset M_i^0$. Since each M_i is ε-thick, up to subsequence we have geometric convergence of $(x_i, M_i) \to (x_\infty, M_\infty)$.

For large enough i we have 2-bi-Lipschitz embeddings

$$\varphi_i : M_i^0 \to M_\infty$$

but they don't necessarily have the same image, so we have yet to arrive at a contradiction.

We need change the image of the above maps in a controlled way. Start by fixing a particular boundary component in each M_i^0 and considering the sequence of maps:

$$S_{g_i} \xrightarrow{\iota_i} \partial M_i^0 \xrightarrow{\varphi_i} M_\infty,$$

where $S_{g_i} \in \mathcal{S}_{g \leq G}$. Up to subsequence we can assume the domains are the same: S_g for some fixed g. Because all the maps are uniformly bi-Lipschitz with images in a compact region, they converge to some uniformly bi-Lipschitz map $\varphi_\infty : S_g \to M_\infty$. Note that because geometric convergence is actually smooth, the maps $\iota_i \circ \varphi_i$ and their limit φ_∞ are all smooth. As such, notice that for large enough i, $\operatorname{im}(\iota_i \circ \varphi_i)$ lies in an arbitrarily small neighborhood of $\operatorname{im}(\varphi_\infty)$ and is transverse to its normal bundle.

Thus for large enough i we can find isotopies $\psi_i : M_\infty \to M_\infty$ taking $\operatorname{im}(\varphi_i \circ \iota_i)$ to $\operatorname{im}(\varphi_\infty)$ by locally stretching in the direction of the normal bundle of $\operatorname{im}(\varphi_\infty)$. By making sure such an isotopy is supported on a slightly larger neighborhood, we can choose the isotopies to be 2-bi-Lipschitz. When we compose the φ_i with associated 2-bi-Lipschitz isotopies described above, we arrive at a sequence of 4-bi-Lipschitz maps on the tail of the sequence M_i^0, but they still don't necessarily have the same image.

We iterate this process by next fixing an unused boundary component in each M_i^0 and repeating the process. (We don't have to worry about distinct boundary components converging in the limit because the Hausdorff distance between any two boundary components of a M_i^0 is uniformly bounded from below.) We do this until we have, up to subsequence, bijections $M_i^0 \to M_\infty^0 \subset M_\infty$, where M_∞^0 is compact. This will happen after a finite number of iterations because we have a uniform bound on the number of boundary components in each M_i^0.

$\square$

The next Corollary follows almost immediately from Steps 1 and 2 in the proof of Theorem V.2. It is also the specific statement we will need to prove Theorem I.1 in §VI.

Corollary V.7. *Given $\varepsilon, c > 0$, there exist constants $p, G, V_{\min}, V_{\max} > 0$ and $L > 1$ such that for all but finitely-many hyperbolic 3-manifolds M satisfying $(*)$ there exists a submanifold $T \subset M$ with the following properties:*

(i) $\mathrm{vol}(T) \geq p\,\mathrm{vol}(M)$.

(ii) *T is homeomorphic to a product region $\Sigma_g \times I$ and $g \leq G$.*

(iii) *$T = T_1 \cup \cdots \cup T_n$ with disjoint interiors.*

(iv) *Finally, each T_i is L-bi-Lipschitz equivalent to $S_g \times [-1, 1]$, where $S_g \in \mathcal{S}$. In particular, there exists $V_{\min}$ and $V_{\max}$ such that $V_{\min} \leq \mathrm{vol}(T_i) \leq V_{\max}$ for every i.*

Proof. Because there are only finitely many components of the tiny manifold $\widetilde{M}$, and they come from a finite family of bi-Lipschitz classes, there is a bound on the number of components of $\partial\widetilde{M}$. In particular, there is an upper bound on the possible number of components of $M - \widetilde{M}$, call it t. Additionally, there is some upper bound on the volume of $\widetilde{M}$, call it V. Suppose $\mathrm{vol}\, M \geq 2V$. Then there exists some component $T \subset M - \widetilde{M}$ such that $\mathrm{vol}\, T \geq \mathrm{vol}\, M/2t$. As we've noted before, there are only finitely-many isometry classes of ε-thick hyperbolic 3-manifolds of of volume less than $2V$.

The rest of the statements follow immediately from the proof of Theorem V.2. $\square$

CHAPTER VI

Proof of Theorem I.1

Recall our main result:

Theorem I.1. *For every $\varepsilon > 0$, $c, k \in \mathbb{N}$ there exists $\Omega(\varepsilon, c, k)$ such that, if M is a closed, ε-thick hyperbolic 3-manifold with* $\operatorname{rank} \pi_1(M) < c$*, then*

$$\frac{1}{\Omega \operatorname{vol}(M)^2} \leq \lambda_k(M) \leq \frac{\Omega}{\operatorname{vol}^2(M)}$$

where $\lambda_k(M)$ is the kth positive eigenvalue of the Laplace operator on M.

Remark. We rely on Corollary V.7 for the following proof. Recall that Corollary V.7 holds for all but finitely many M satisfying $(*)$. It suffices to prove the theorem in this context because for finitely-many M we can always find some Ω to satisfy the inequality by taking $\max\{\lambda_k(M_\alpha) \operatorname{vol}^2(M_\alpha), \delta^{-1}\}$ over a finite index set $\alpha \in A$, where δ comes from Schoen's lower bound (see Theorem III.6).

Proof. As mentioned in the introduction, the lower bound is due to Schoen [29], because $\lambda_k \geq \lambda_1$.

To prove the upper bound, we use the "max" half of the Minimax Theorem. That is, for any set of functions $f_0, \ldots, f_k : M \to \mathbb{R}$ whose supports pairwise intersect with on zero-volume sets, we have $\lambda_k \leq \max\{R(f_i)\}_{i=0}^k$, where $R(f) = \frac{\int_M ||\nabla f||^2}{\int_M f^2}$. We will define test functions piecewise on the decomposition $T = \bigcup_{i=1}^n T_i$ from Corollary V.7.

Let $\varphi : [-1, 1] \to [0, 1]$ be a smooth function such that

(i) $\varphi(-1) = 0$,

(ii) $\varphi(1) = 1$,

(iii) and φ is constant in some neighborhood of -1 and 1.

Now define $\Phi : S_g \times [-1, 1] \to [0, 1]$ by $\Phi(x, t) = \varphi(t)$. We have L-bi-Lipschitz embeddings $(f_i)^{-1} : T_i \to S_g \times [-1, 1]$. To shorten notation, let $F_i = (f_i)^{-1}$. Thus $\Phi \circ F_i$ is a function $T_i \to [0, 1]$.

Recall that $\lfloor \star \rfloor$ denotes "the integer part of $\star$." Let $m = \left\lfloor \frac{n}{2(k+1)} \right\rfloor$. Note that T can be decomposed more coarsely into $k + 1$ pieces (which will be the supports of our $k + 1$ test functions), each consisting of $2m$ consecutive T_i, i.e.,

$$P_0 = T_1 \cup \cdots \cup T_{2m}, \ldots, P_k = T_{2km+1} \cup \cdots \cup T_{2m(k+1)}$$

and a possibly empty piece $P_{\text{extra}} = T_{2m(k+1)+1} \cup \cdots \cup T_n$. We will describe functions g_κ supported on P_κ, $0 \le \kappa \le k$, and then bound their Rayleigh quotients.

Roughly speaking, each g_κ will grow to m and decrease back down to 0 by increasing (or decreasing) by one on each T_i using an appropriate translation of $F_i \circ \Phi$. More precisely, for $0 \le \kappa \le k$,

$$g_\kappa(x) = \begin{cases} \Phi \circ F_{2m\kappa+j}(x) + j & x \in T_{2m\kappa+j},\ j \in [1, m] \\ -\Phi \circ F_{2m\kappa+j}(x) + 2m - j + 1 & x \in T_{4m\kappa+j},\ j \in [m+1, 2m] \end{cases}$$

Before getting too bogged down in this definition, look at Figure VI. Also realize that all the g_κ are "copies" of g_0 shifted over.

Recall that our goal is to bound $R(g_\kappa) = \frac{\int_M ||\nabla g_\kappa||^2 dM}{\int_M g_\kappa^2 dM}$ for each $0 \le \kappa \le k$. Note

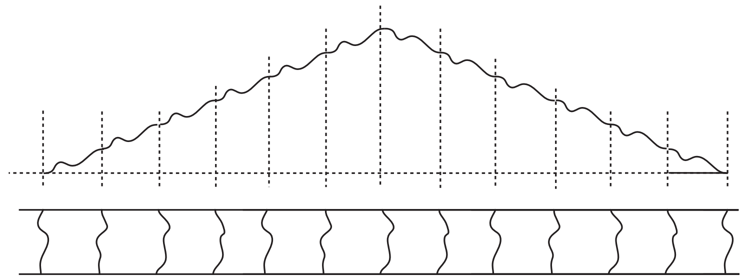

Figure 6.1: An example of g_κ when $m = 6$.

that where g_κ is non-constant the following holds by the chain rule:

$$\begin{aligned}\int_{T_i} ||\nabla g_\kappa||^2 \, dM &= \int_{F_i(T_i)} \left|\left|\nabla (g_\kappa \circ F_i^{-1})\right|\right|^2 (F_{i*} dT_i) \\ &= \int_{S_g \times [-1,1]} ||\nabla \Phi||^2 \ (F_{i*} dT_i) \\ &\leq L^3 \int_{S_g \times [-1,1]} ||\nabla \Phi||^2 \ d(S_g \times [-1,1])\end{aligned}$$

where L^3 factor comes from the fact that each F_i is an L-biLipschitz embedding and our volume forms are 3-forms. Thus

$$\int_M ||\nabla g_\kappa||^2 \, dM = \int_{P_\kappa} ||\nabla g_\kappa||^2 \, dM \leq 2mL^3 \int_{S_g \times [-1,1]} ||\nabla \Phi||^2 \ d(S_g \times [-1,1])$$

Also note that

$$\int_M g_\kappa^2 dM = \int_T g_\kappa^2 dM \geq 2V_{\min} \sum_{i=1}^{m} (m-1)^2 \geq V_{\min} m^3/6$$

by definition of g_κ.

Now we can calculate:

$$\begin{aligned} R(g_\kappa) &= \frac{\int_M ||\nabla g_\kappa||^2 \, dM}{\int_M g_\kappa^2 dM} \\ &\leq \frac{\frac{m}{2} L^3 \int_{S_g \times [-1,1]} ||\nabla \Phi||^2 \ d(S_g \times [-1,1])}{\frac{V_{\min} m^3}{6}} \\ &\leq \frac{K'}{m^2} \end{aligned}$$

where K' depends only on ε, k, c. Lastly note that $m \geq \frac{n}{2k}$, $n \geq \mathrm{vol}(T)/V_{\max}$, and by Corollary V.7 there exists p such that $\mathrm{vol}(T) = p\,\mathrm{vol}(M)$. Thus

$$m \geq \frac{p\,\mathrm{vol}\,M}{2kV_{\max}}$$

With that, Theorem I.1 is proved. $\square$

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